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CAPITAL ACCUMULATION IN A STOCHASTIC DECENTRALIZED ECONOMY*

by

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1. Introduction

One goal of a dynamic general equilibrium theory is to explain how the rate of aggregate capital accumulation and the equilibrium values of assets are determined in a decentralized market economy. Whether such a model assumes uncertainty is present or not, it will combine a theory of the consumer's problem of intertemporal allocation of wealth between consumption and saving with a theory of production. There are several ways to approach this problem. The Arrow [1] and Debreu [6] theory is the most general and abstract model. While its approach is very valuable in yielding theoretical insight, it is difficult to apply in empirical study. Several important papers, such as those by Diamond [8] and Leland [13], discuss security markets and production sectors in versions of the Arrow-Debreu framework. These are two-period models with uncertain production in the second period.

Another way to develop an equilibrium model is to begin with a less abstract and more detailed description of consumer behavior and the technology. The Cass and Yaari [4] paper, an extension of work by Diamond [7] and Samuelson [21], is a major contribution to the effort to derive a description of aggregate dynamic capital accumulation under certainty from a life-cycle theory of consumer behavior and a neoclassical technology.

Under uncertainty, most attention has been focused on problems with a linear technology. Phelps [20], for example, considers the discrete-time optimal stochastic consumption-saving problem, while Hakansson [10], Leland [12], and Samuelson [22] consider the discrete-time consumption-portfolio problem. Merton [15] examines the latter problem in continuous-time. Many of these models may be interpreted as either the portfolio problem faced by an investor in a decentralized market or as the centralized stochastic Ramsey problem for an economy with several linear technologies. The most significant extension of the linear models to the pricing of assets in a decentralized economy is the intertemporal capital asset pricing model of Merton [16]. In contrast there are not nearly so many papers dealing with nonlinear technologies. Among the important studies of stochastic growth models are the papers by Bourguignon [2], Brock and Mirman [3], Merton [17], and Mirrlees [9].

In this paper we study an economy where decentralized decision-making by many consumers, each solving a continuous-time consumption-portfolio problem, determines an aggregate saving policy. Suppose there is a single good which may be consumed or combined with labor in a neoclassical production process to produce more of the good. Production is carried on by many firms operating under perfect competition. The change in the labor force over time is described by a stochastic process. The two assets for consumer investment are the capital good and claims to future labor income, which is denoted human capital. While capital is a riskless asset, the stochastic labor force growth, with the consequent uncertainty in future wages, implies human capital is a risky asset. A competitive market is described where claims on capital and human capital are traded, and the value of human capital is determined.

2. Population Dynamics

We begin with a branching process model of changes in the population size $L(t)$.[1] After obtaining expressions for the instantaneous conditional expected change in the population and the instantaneous conditional variance, we present the continuous-time version of the discrete-time process as a Poisson-driven stochastic differential equation.[2] Assume h is the length of time between generations.[3] Further assume the expected value of the number of offspring of the i^{th} individual alive at time t is a constant $(n + \phi\lambda)h$, the same for all individuals of all generations; and random variable deviations from the mean may be written as $\phi\sqrt{\lambda}\eta(t;h) + \nu\varepsilon_i(t;h)$ where the first term reflects systematic random effects common to everyone and the second term represents random effects specific to the i^{th} individual. Also, assume ϕ is positive. Then the total number of births between t and $t+h$ is $(n + \phi\lambda)Lh + \phi\sqrt{\lambda}L\eta(t;h) + \sum_{i=1}^{L}\nu\varepsilon_i(t;h) \equiv b(t;h)$.

Also define for each person a random variable $d_i(t;h)$. Let $d_i(t;h) = 1$ if the i^{th} individual dies between t and $t+h$ and 0 otherwise. Suppose lifetimes are independent, the expected value of $d_i(t)$ is mh, and the variance is θh for all individuals. Then the total number of deaths for a population size L may be expressed as $\sum_{i=1}^{L} d_i(t;h) = mLh + \theta\sqrt{L}\,\xi(t;h) \equiv d(t;h)$, where ξ is a random variable with zero mean. Finally, suppose births and deaths are independent and all random variables are serially independent.

Therefore, conditional on $L(t) = L$, the total change in population may be written

$$
\begin{aligned}
L(t+h) - L(t) &= b(t;h) - d(t;h) \\
&= (n-m+\phi\lambda)Lh + \phi\sqrt{\lambda}L\eta(t;h) + \sum_{i=1}^{L} \nu\varepsilon_i(t;h) - \theta\sqrt{L}\xi(t;h), \qquad (1)
\end{aligned}
$$

where n, m, λ, ϕ, ν, θ are constant; $E_t(\eta) = E_t(\varepsilon_i) = E_t(\xi) = 0$; $E_t(\eta^2) = E_t(\varepsilon_i^2) = E_t(\xi_t^2) = h$; $E_t(\eta\varepsilon_i) = E_t(\xi\varepsilon_i) = E_t(\eta\varepsilon_i) = E_t(\varepsilon_i\varepsilon_j) = 0$, $i \neq j$; and "E_t" is the conditional expectation operator, conditional on knowledge of all events which have occurred as of time t.

From (1) we obtain the conditional expected change in the population size and the conditional variance:

$$E[L(t+h) - L(t)|L(t) = L] = (n-m+\phi\lambda)Lh \tag{2}$$

and

$$\mathrm{Var}[L(t+h)-L(t)|L(t) = L] = (\phi^2\lambda + \frac{\nu^2}{L} + \frac{\theta^2}{L})L^2h. \tag{3}$$

For L very large we may approximate the conditional variance as

$$\mathrm{Var}[L(t+h)-L(t)|L(t) = L] = \phi^2\lambda L^2h. \tag{4}$$

This approximation implies only the systematic component of uncertainty in births is regarded as significant. This stochastic process may be described as a process such that if a random event occurs, then the population increases by ϕL, where L is the current population. Moreover, the times between random events are independent and identically distributed random variables.

A continuous-time model consistent with our assumptions for the events is the Poisson process. If λ is the mean number of events per unit time, then the Poisson process has the properties that for all $h > 0$,

$$\begin{aligned} &\text{prob}\{\text{exactly one event occurs in } (t,t+h)\} = \lambda h + o(h) \\ &\text{prob}\{\text{no event occurs in } (t,t+h)\} = 1-\lambda h + o(h) \\ &\text{prob}\{\text{more than one event occurs in } (t,t+h)\} = o(h); \end{aligned} \tag{5}$$

where a function f is $o(h)$ if $\lim_{h\to 0} \frac{f(h)}{h} = 0$.

Our reasons for using the continuous-time model are that it is a reasonable description of the population dynamics and it is analytically more tractable than the discrete-time process. In particular, subsequent sections of this paper make extensive use of Merton's powerful continuous-time results in dynamic portfolio theory.

The appropriate mathematical tools for treating Poisson-driven stochastic processes are results from a theory of stochastic integral and differential equations analogous to the theory of Ito and McKean for diffusion processes.[4] Stochastic differential equations are important because they provide a convenient characterization of many continuous-time stochastic processes. The stochastic differential equation corresponding to the population process is

$$dL = (n-m)Ldt + \phi L dq, \tag{6}$$

where dq is a Poisson process increment and $(n-m+\phi\lambda)L$ and $\lambda\phi^2L^2$ are the instantaneous mean and variance per unit time, respectively. Since stochastic differential equations do not obey the rules of ordinary calculus, special techniques must be used for integration and differentiation. Applying the version of Itô's lemma[5] for jump processes to integrate (6), we find $L(t)$ may be written as the log-Poisson process

$$L(t) = L(0)e^{(n-m)t}\cdot(\phi+1)^{q(t)}, \tag{7}$$

where $q(t)$ is the Poisson process with mean rate λ. We assume $n-m > 0$. This and $\phi > 0$ are sufficient to imply $L(t)$ becomes infinitely large with probability one. Recall one condition for the accuracy of our approximation in (4) is large L.

3. Technology and Production

The technology for transforming capital and labor inputs into output is described by a concave, constant returns to scale production function $F(K,L)$, where $K(t)$ and $L(t)$ are the quantities of capital and labor available at time t.[6] Over a small interval h, output of the firms is given by

$$K(t+h) - K(t) = F(K,L)h. \tag{8}$$

Thus, although the stochastic nature of $L(t)$ implies both $K(t)$ and $L(t)$ are stochastic processes, the output at t+h is completely certain, conditional on knowledge of $K(t)$ and $L(t)$.[7] This means the firms' decisions are characterized by the deterministic neoclassical theory of the firm. Define

$$k \equiv K/L \equiv \text{capital to labor ratio}$$

$$f(k) \equiv F(K,L)/L \equiv F(K/L,1) \equiv \text{gross per capita output.}$$

Under the assumptions that profit maximizing firms hire the services of labor through perfectly competitive factor markets and that the capital good is <u>numeraire</u>, the labor market is in equilibrium at all times only if the wage rate over the time interval h is equal to $(f(k)-kf'(k))h$. This implies capital's share of the output on a per unit basis is $f'(k)h$. Therefore, since the preceding discussion is valid for all positive h, we may define an interest rate and a wage rate which satisfy

$$r(k) = f'(k) \tag{9}$$

$$w(k) = f(k) - kf'(k). \tag{10}$$

Thus, there is no instantaneous uncertainty in the production sector. The portions of output distributed to capital and labor over the next very short "period" are known. The future values of the interest rate and wage rate are unknown, however. The time evolution of r and w depends on the stochastic process for k.

To derive the stochastic differential equation description of the k process, we begin with the aggregate capital accumulation equation

$$\dot{K}(t) = F(K,L) - \delta K(t) - C(t), \tag{11}$$

where δ is the rate of depreciation of the capital good and C(t) is aggregate consumption. To write the capital accumulation equation in per capita terms, we use the population dynamics equation (6) and Ito's lemma for Poisson-driven processes. Thus, the stochastic differential equation describing changes in the capital to labor ratio is

$$dk = (f(k) - \beta k - c(t))dt - \sigma k dq, \tag{12}$$

where $\sigma = \phi/(1+\phi)$, $\beta = n-m+\delta$, and c(t) is per capita consumption. In later sections of this paper, we discuss how C(t) and c(t) are determined.

It is apparent from (11) and (12) that while the paths of both K(t) and k(t) are stochastic, these paths are very different. The K(t) process is random because it is a function of the L(t) process. Its derivative exists and thus competitive factor shares are determined with certainty as discussed previously. In contrast, the increments of k(t) are stochastic. That is, given the information at time t, namely K(t) and L(t), the infinitesimal change in k(t) is random and the time path is not differentiable. In particular, jumps in k(t) occur at the same time as the population process L(t) jumps.

Thus, the stochastic process for k is a Poisson-driven jump process. If f(k) is a sufficiently differentiable function, then r and w will also be jump processes. Later we will calculate the stochastic differential of r for a specific f(k).

4. The Financial Markets

To determine how uncertainty affects the decisions of households, consider the position of an individual in the economy. At birth he owns the stream of wages he will earn during his lifetime.[8] Ownership of this wage stream defines an intertemporal set of consumption possibilities. Although one feasible consumption plan is simply to consume at the wage rate, there are two fundamental reasons why this is not the preferred plan of most individuals. First, even if future values of wages were certain, there is little reason to believe that individuals of diverse wealths, ages, and other characteristics prefer to consume at the wage rate. Second, in an economy with uncertain future wages and uncertain lifetime, owning the lifetime earnings stream is tantamount to owning a risky asset. Though some people may prefer to hold all their wealth in one risky asset, reasonable assumptions of risk aversion imply a preference to reduce exposure to any one source of risk by distributing wealth among several assets.

Therefore, it is reasonable to assume there is a financial sector in the economy to provide a household with the means to distribute its wealth among several assets and to determine an optimal intertemporal consumption plan consistent with the constraint that wealth always be nonnegative. To characterize the financial market, we specify the assets traded and each asset's return structure.[9] We recall the definition of the return from holding an asset over the period to to t+h, when the value per share or unit is V(t), is

$$\frac{V(t+h) - V(t) + Y(t)h}{V(t)}, \tag{13}$$

where $V(t+h) - V(t)$ is the capital gain (or loss) portion of the return and $Y(t)$ is the instantaneous rate of noncapital gains payments, such as dividends.

Although all physical capital is owned by households, we assume the production firms hold the capital and issue shares to individuals for the capital held. Furthermore, let these shares be traded in a secondary securities market.[10] The return to these shares is determined by the interest rate (9) and the depreciation rate. If each share represents one unit of physical capital, then during the period h, each share is entitled to a "dividend" of $f'(k)h - \delta h = (r-\delta)h$ units of capital. To pay this the firms split the shares so that one share at time t becomes $1+(r-\delta)h$ shares at t+h. Because these noncapital gains distributions equal the rate of return on capital, the equilibrium price per share must be one at all times. Thus, the rate of return per share is

$$r(k) - \delta, \tag{14}$$

and shares represent an asset equivalent to physical capital.

Because (14) is completely certain at time t given the information $k(t)$, the return to capital shares is known over the next short "period". In this sense a capital share is an instantaneously riskless asset, although it is not completely riskless since future values of $k(t)$ are uncertain. Hence investing in capital shares is similar to investing in short term Treasury bills where the current rate of return is known, but future rates are uncertain.

Although we have shown that individuals are indifferent between holding physical capital and capital shares, we must still explain how consumption is accomplished if only firms hold capital. It is not difficult to see that a necessary condition for equilibrium in the real and financial markets is that firms purchase shares at a rate equal to the aggregate consumption rate. If they did not, there would be an excess supply at the price one and to equilibrate the financial market the price would fall. When that happened, the rate of return the firm could earn by buying its own shares would exceed the real rate given by the production technology. Hence profit maximizing firms would exchange capital for shares. Of course this process would continue only until the share price reached one. Similarly if the share price exceeded one, firms would sell new shares until the price reached one. Therefore, the only equilibrium price is one and all households may realize their optimal consumptions plans by selling capital shares.

To exchange a claim to wages for capital shares, an individual must hold a well-defined asset representing his human capital.[11] Suppose when someone is born, he signs a contract with a financial intermediary, hereafter called the mutual fund, giving the fund a claim on all the individual's lifetime earnings in return for a share of the fund. Thus, individuals own human capital through ownership of the fund. Moreover, the fund's shares are traded in the secondary market along with capital shares. Though capital is instantaneously riskless, we will show the fund's shares are risky. In other words, the fund's shares are similar to common stocks.

The value of the mutual fund at time t is $V(t) = N(t)P(t)$, where $N(t)$ is the number of shares outstanding and $P(t)$ is the price per share. If we assume all individuals exchange their wages for shares of the fund, then the fund receives an inflow of capital from wages at the rate $L(t)w(k)$. Since the choice of $N(t)$ is arbitrary, it is convenient to set $N(t) = L(t)$. Then the value of the fund is $L(t)P(t)$, where $P(t)$ is now defined to be the per capita value of human capital. Note that each share receives dividends at the rate $w(k)$.

To complete the description of the return structure, we must analyze the effects of births and deaths on the supply of shares. As before, let $b(t;h)$ and $d(t;h)$ denote the number of births and deaths between t and $t+h$. Let each individual born during this time interval receive one share of the fund at $t+h$. In other words, each receives a claim worth $P(t+h)$. In addition, assume that in order to set the supply of shares at $L(t+h)$, the fund reverse splits the shares so that each share existing at t becomes $1-d(t;h)/L(t)$ shares at $t+h$. This means each share receives a dividend at $t+h$ of $w(k)h-P(t+h)d(t;h)/L(t)$.[12] Therefore, the rate of return during the period t to $t+h$ is

$$\frac{P(t+h) - P(t) + w(k)h - P(t+h)d(t;h)/L(t)}{P(t)} . \tag{15}$$

Since lifetimes are independent, it is reasonable to approximate $d(t;h)$ as mLh.[13] In the limit as h approaches 0, (15) becomes the continuous-time version

$$\frac{dP + w(k)dt - P(t)mdt}{P(t)} . \tag{16}$$

Figure 1 illustrates the structure of the economy described in this section. We note the financial market provides significant services to the households. For without capital shares, individuals would have to store capital and then provide it for production. Without the mutual funds, households would have to establish many separate claims among themselves if they wanted to exchange claims to future labor income for capital.

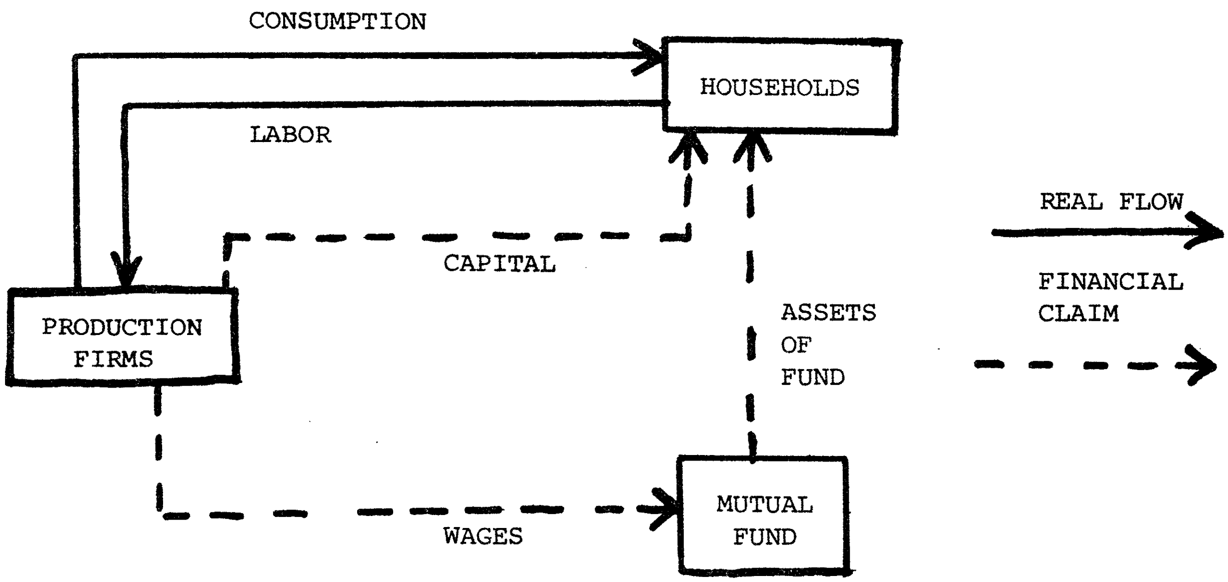

Figure 1

To find the per capita value of human capital, we need to determine P as a function of the relevant information. Relevant information in an economy with a general preference structure includes the values of all variables influencing investor demand for the risky asset. Among these are individual wealths, r, w, k, and the stochastic processes related to individual lifetimes. This problem is simplified somewhat if it is assumed all this information may be described by a set of Markovian state variables satisfying stochastic differential equations. Then P is a function of the state variables and satisfies a stochastic differential equation. Analytically this problem is still very difficult. We will discuss conditions under which the per capita value of human capital may be written as a function of k, i.e. $P = P(k)$. Then the generalized version of Itô's lemma may be used to write the stochastic differential dP in terms of k. This in turn implies the rate of return satisfies

$$\frac{dP + w(k)dt - Pmdt}{P} = \alpha_H dt - \sigma_H dq_H, \tag{17}$$

where α_H and σ_H may be functions of k and q_H is a Poisson process. Since population change is the only source of uncertainty, the q_H process jumps at exactly the same instants as the population. We return to the valuation problem after determining the optimal portfolio and consumption rules for the individual, under the assumption that the rate of return is characterized by a stochastic differential of the form (17).

5. The Consumption-Portfolio Problem

We assume households have homogeneous and rational expectations. Homogeneous expectations means they all have the same subjective probability distributions, while rational expectations means the stochastic processes which consumers believe describe changes in the investment opportunity set are the correct processes. That is, no statistical test of any kind on the observed past history of the economy would indicate beliefs should be changed. Hence, there is no need for adaptive revision of expectations.

The investment opportunity set is described by $r-\delta$ and (17). There are two assets, namely riskless capital shares and risky shares of the mutual fund. Consequently, the i^{th} consumer's wealth at time t is $W^i(t) = N_1^i(t) + N_2^i(t)P(t)$, where $N_1^i(t)$ is the number of capital shares owned and $N_2^i(t)$ is the number of mutual fund shares owned. By setting $\omega^i = N_2^i P/W^i$, the fraction of wealth invested in the fund, we may derive the budget equation

$$dW^i = [(\alpha_H - (r-\delta))\omega^i W^i + (r-\delta)W^i - c^i(t)]dt - \sigma_H \omega^i W^i dq_H, \tag{18}$$

where $c^i(t)$ is the rate of consumption.[14]

To determine the optimal rate of consumption and the optimal portfolio, an individual i born at time τ solves

$$\text{Max } E_\tau\left[\int_\tau^{\tau+T_\tau^i} U^i(c^i(s),\ s,\ \tau)ds + B(W^i(\tau + T_\tau^i\ ))\right] \tag{19}$$

subject to (18) and $W^i(\tau) = W^i$, the initial wealth; and where U^i is a Von Neumann-Morgenstern utility function and "E_τ" is the conditional expectation operator, conditional on the current value of wealth and any other state variables necessary to describe changes in the investment opportunity set and to determine the distribution of T_τ^i, the individual's random lifetime. If the vector of state variables is denoted $S(t)$, then we may solve (19) by stochastic dynamic programming to find the optimal consumption and portfolio rules as functions of $S(t)$ and t.

To make the analysis tractable, we impose several additional assumptions. Suppose at time τ individual i acts so as to

$$\text{Max } E_\tau\left[\int_\tau^\infty e^{-\rho(s-\tau)}\log(c^i(s))ds\right] \tag{20}$$

subject to (18) and $W(\tau) = W_\tau^i$, wealth at τ.[15] In this problem the state variables do not include stochastic processes specific to the individual except for wealth. Consequently the relevant state variables for (20) are wealth and the aggregate quantities r, w, k, α_H, and σ_H. From (9) and (10) we see r and w are functions of k. Thus, it is reasonable to assume k is a sufficient statistic for α_H and σ_H. In the next section we solve for $c(t)$, $P(t)$, α_H, and σ_H as functions of k and thus verify the assumption k is sufficient.

If we define

$$J(W^i,k,t) = \underset{c^i,\omega^i}{\text{Max}} \left[E_t \int_t^\infty e^{-\rho s}\log(c^i(s))ds\right], \tag{21}$$

then the next theorem characterizes the optimal c^i and ω^i.[16]

Theorem 1: If W^i and k satisfy (18) and (12), respectively, and there exist c_*^i and ω_*^i which satisfy at each t

$$\begin{aligned} 0 = \underset{c^i,\omega^i}{\text{Max}} \Big[\log c^i - \rho I + \frac{\partial I}{\partial W^i} [(\alpha_H - (r-\delta))\omega^i W^i \\ + (r-\delta)W^i - c^i] + \frac{\partial I}{\partial k} [f(k) - \beta k - c(t)] \\ + \lambda[I(W^i - \sigma_H\omega^i W^i, k - \sigma k) - I(W^i, k)]\Big] \end{aligned} \tag{22}$$

with $I(W^i, k) \equiv e^{\rho t}J(W^i,k,t)$, then c_*^i and ω_*^i are optimal for the objective (20).

This theorem allows us to derive a partial differential-difference equation for I. The first order conditions derived from (22) are

$$c_*^i = 1/I_{W^i} \tag{23}$$

and

$$0 = I_{W^i}[\alpha_H - (r-\delta)] - \lambda\sigma_H I_{W^i}(W^i - \sigma_H\omega_*^i W^i, k - \sigma k), \tag{24}$$

where the subscript indicates partial differentiation. Equation (24) defines ω_*^i as an implicit function which we may write as

$$\omega_*^i = H(W^i, k). \tag{25}$$

Substituting (25) and (23) into (22) yields the equation

$$\begin{aligned} 0 = -\log(I_{W^i}) - \rho I + I_{W^i}[(\alpha_H - (r-\delta))HW^i + (r-\delta)W^i - 1/I_{W^i}] \\ + I_k[f(k) - \beta k - c(t)] + \lambda[I(W^i - \sigma_H HW^i, k - \sigma k) - I(W^i, k)], \end{aligned} \tag{26}$$

subject to the boundary condition $\lim_{t\to\infty} [e^{-\rho t} I(W^i,k)] = 0$. This condition is satisfied if $\rho > 0$. A solution to (26) is

$$I(W^i, k) = \frac{1}{\rho} \log W^i + a(k), \tag{27}$$

where $a(k)$ satisfies the equation

$$0 = \frac{da(k)}{dk} [f(k) - \beta k - c(k)] - (\rho+\lambda)a(k) + \lambda a(k-\sigma k) + \log \rho + \frac{r-\delta-\rho}{\rho} + \frac{\alpha_H-(r-\delta)}{\rho\sigma_H}\left(1 - \frac{\lambda\sigma_H}{\alpha_H-(r-\delta)}\right) + \frac{\lambda}{\rho} \log \frac{\lambda\sigma_H}{\alpha_H-(r-\delta)} \ . \tag{28}$$

This implies the optimal rate of consumption and portfolio demand for individual i satisfy

$$c_*^i = \rho W^i \tag{29}$$

and

$$\omega_* W^i = \frac{1}{\sigma_H} \left(1 - \frac{\lambda\sigma_H}{\alpha_H-(r-\delta)}\right) W^i. \tag{30}$$

It is important to note (29) and (30) are linear in W^i,[17] that c_*^i depends only the state variable W^i, and that the fraction ω_* of wealth invested in the risky asset depends on the state variable k only through α_H, σ_H, and r. Though the linearity in wealth of c_*^i and $\omega_* W^i$ is a characteristic property of the class of utility functions displaying hyperbolic absolute risk aversion (HARA), the particularly simple form of (29) and (30) occurs only for the log utility function.[18]

6. Value of the Risky Asset

The equilibrium value of human capital is determined by setting the aggregate demand for the risky asset equal to its total value. It is assumed the market is in equilibrium at every instant; thus, the equilibrium relationship must hold at all times. From (30), the aggregate demand for the risky asset is

$$\sum_{i=1}^{N} \frac{1}{\sigma_H} \left(1 - \frac{\lambda\sigma_H}{\alpha_H - (r-\delta)}\right) W^i = \frac{1}{\sigma_H}\left(1 - \frac{\lambda\sigma_H}{\alpha_H - (r-\delta)}\right) \sum_{i=1}^{N} W^i$$

$$= \frac{1}{\sigma_H}\left(1 - \frac{\lambda\sigma_H}{\alpha_H - (r-\delta)}\right) (K + PL), \tag{31}$$

where N is the total number of households (not necessarily equal to L). Since the total value of the risky asset is PL, we obtain

$$\frac{1}{\sigma_H}\left(1 - \frac{\lambda\sigma_H}{\alpha_H - (r-\delta)}\right) = \frac{P}{k + P}. \tag{32}$$

To use the rate of return (17) to determine α_H and σ_H, we must know the state variables upon which P depends so that the generalized Itô's lemma may be applied to calculate dP. Examination of (32) and the capital accumulation equation (12) leads to the conclusion that k is sufficient to specify P, i.e. P = P(k).[19] Applying the differentiation rule to P(k) yields

$$dP = P'(k)(f(k) - \beta k - \rho(k+P))dt + (P(k-\sigma k) - P(k))dq_H, \tag{33}$$

where $P' \equiv dP/dk$. Consequently, (17) and (33) imply

$$\alpha_H = \frac{1}{P(k)} [P'(k)(f(k) - \beta k - \rho(k+P(k)) + f(k) - kf'(k) - mP(k)] \tag{34}$$

and

$$\sigma_H = -\left[\frac{P(k-\sigma k) - P(k)}{P(k)}\right]. \tag{35}$$

Substituting (34) and (35) into the equilibrium equation (32) gives

$$\frac{P(k)}{k+P(k)} = \frac{P(k)}{P(k)-P(k-\sigma k)} - \frac{\lambda P(k)}{P'(k)(f(k)-\beta k-\rho(k+P(k))+f'(k)-kf'(k)-mP(k)-(r-\delta)P(k)}. \tag{36}$$

This may be rearranged as

$$[P'(k)(f(k)-\beta k-\rho(k+P(k))+f(k)-kf'(k)-mP(k)-(r-\delta)P(k)][k+P(k-\sigma k)]$$
$$-\lambda[k+P(k)][P(k)-P(k-\sigma k)] = 0. \quad (37)$$

Thus, the equilibrium value of the risky asset $P(k)$ satisfies the nonlinear differential-difference equation (37) on the interval $0 \leq k < \infty$. Because $f(0) = 0$ and prices must be nonnegative, we have the boundary conditions $P(0) = 0$ and $P(k) \geq 0$. It is not clear, however, that these conditions determine a unique solution for a given technology $f(k)$. In the next section, the set of technologies for which (37) admits a solution linear in k is characterized.

7. An Example

The following theorem provides the motivation for studying the example in this section.

Theorem 2: The linear function $P(k) = \gamma k$ solves (37) if and only if $f(k) = (A-B\log k)k$ where A and B are constants. Furthermore, if $B > 0$, then there is exactly one solution with $\gamma > 0$.

Pf:If: When $f(k) = (A-B\log k)k$ and γk are substituted into (37), the differential-difference equation becomes the algebraic equation

$$0 = -\rho(1-\sigma)\gamma^3 + [(1-\sigma)(-\beta-\rho-m+B)-\lambda\sigma]\gamma^2$$
$$+ [-\beta-m-\rho + B + B(1-\sigma)-\lambda\sigma]\gamma + B. \quad (38)$$

This cubic equation has three roots and these provide three linear solutions to (37). To prove $B > 0$ implies there is exactly one positive root of (38), note that the assumptions $\lambda > 0$, $0 < \sigma < 1$ exclude the possibility that the coefficients of γ^2 and γ are positive and negative, respectively. Then by

Descartes' rule of signs, (38) has exactly one positive root.

Only if: Assume $P(k) = \gamma k$ and substitute into (37). This gives

$$[(1+\gamma)f - (1+\gamma)kf' - \beta\gamma k - \rho\gamma(1+\gamma)k - m\gamma k][1 + \gamma(1-\sigma)]$$
$$- \lambda[1+\gamma]k\gamma\sigma = 0, \tag{39}$$

which implies

$$f - kf' + ak = 0, \tag{40}$$

where

$$a = -\frac{(\beta\gamma+\rho\gamma(1+\gamma)+m\gamma)(1+\gamma(1-\sigma))+\lambda\gamma\sigma(1+\gamma)}{(1+\gamma)(1+\gamma(1-\sigma))}. \tag{41}$$

From (40), we have

$$(k^{-1}f)' \equiv -k^{-2}(f-kf') = ak^{-1}. \tag{42}$$

Integration of (42) yields

$$f(k) = (a\log k + b)k, \tag{43}$$

where b is a constant. Q.E.D.

Assume $A > 0$ and $B > 0$ so $f(k) = (A - B \log k)k$ is a strictly concave production function.[20] The interest rate and wage rate satisfy

$$r(k) = A - B - B \log k \tag{44}$$

$$w(k) = Bk. \tag{45}$$

For this technology the per capita value of human capital is γk, where γ is the unique positive solution of (38). Notice $P(0) = 0$ and $P(k) \geq 0$ for $0 \leq k < \infty$. Though it is possible to solve the cubic equation, the solution provides little more insight than is gained by performing a comparative statics analysis of the function defined by $\gamma \equiv \gamma(B, \rho, n, \sigma, \lambda)$. To begin, we write the equilibrium relation (32) for this linear case as

$$0 = -\frac{1}{\sigma} + \frac{\lambda}{B-n-\rho-\rho\gamma+\frac{B}{\gamma}} + \frac{\gamma}{1+\gamma}. \tag{46}$$

The objective of the comparative statics analysis is to determine the algebraic sign of the partial derivatives of γ with respect to each of its arguments. Define γ_i as the partial derivative of γ with respect to argument i. For notational convenience, also let

$$Y \equiv \left[\frac{\lambda(\rho+B/\gamma^2)}{(B-n-\rho-\rho\gamma+\frac{B}{\gamma})^2} + \frac{1}{(1+\gamma)^2}\right]^{-1} \tag{47}$$

$$Z \equiv B-n-\rho-\rho\gamma+\frac{B}{\gamma}\,. \tag{48}$$

By implicitly differentiating (46), we obtain

$$\begin{aligned}
&\gamma_1 = \frac{Y}{Z^2}\,\lambda\,\frac{\gamma+1}{\gamma} > 0 && \gamma_2 = -\frac{Y}{Z^2}\,\lambda(1+\gamma) < 0\\
&\gamma_3 = -\frac{Y}{Z^2}\,\lambda < 0 && \gamma_4 = -\frac{Y}{\sigma^2} < 0\\
&\gamma_5 = -\frac{Y}{Z} < 0.^{21}
\end{aligned} \tag{49}$$

The results are not surprising. Increasing B is equivalent to increasing the wage rate for all values of k and hence the value of future wages is increased. Larger ρ implies greater impatience to consume and a lower expected return on the risky asset with the same variance. Increasing the birthrate n again implies smaller expected return with the same variance. The parameters σ and λ measure the size and mean frequency, respectively, of the random jumps in population. When a random event occurs, the population always increases, k decreases, and therefore increasing σ or λ will lower the value of a claim to future wages.

The capital accumulation equation for this example is

$$dk = [(A - B\log k)k - \beta k - \rho(1+\gamma)k]dt - \sigma k dq. \tag{50}$$

Using the generalized Itô's lemma for Poisson-driven processes , we calculate

$$d(\log k) = [A-\beta- \rho(1+\gamma) - B \log k]dt - \psi dq, \tag{51}$$

where $\psi \equiv -\log(1-\sigma)$. From (44) and (51), we deduce the dynamics of the interest rate r:

$$dr = B[-B + \beta + \rho(1+\gamma) - r]dt + B\psi dq. \tag{52}$$

The r process is Markov; it has both continuous and jump components.[22] Integration of (52) gives an expression for r(t) conditional on r(τ):

$$r(t)-r(\tau) = [-B + \beta + \rho(1+\gamma) - r(\tau)][1 - e^{-B(t-\tau)}] + B\psi e^{-Bt} \int_\tau^t e^{Bs}dq. \tag{53}$$

Figure 2 shows a typical sample path of (53).

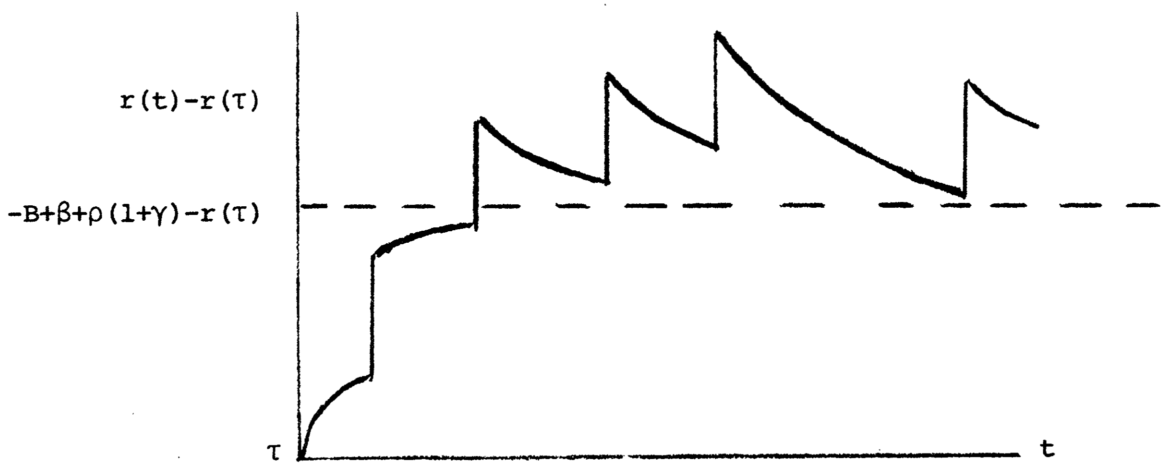

Figure 2

From (53) we may compute the moments

$$E[r(t)-r(\tau)|r(\tau)] = [-B + \beta + \rho(1+\gamma) - r(\tau) + \lambda\psi](1-e^{-B(t-\tau)}) \tag{54}$$

$$\mathrm{Var}[r(t)-r(\tau)|r(\tau)] = \frac{\lambda B\psi^2}{2} (1-e^{-2B(t-\tau)}). \tag{55}$$

Therefore, as t becomes very large, the first two moments converge. To show the whole distribution of r(t) converges to a steady-state, consider (53). The first term on the right side of the equality clearly converges to a constant. The second term is stochastic and has characteristic function

$$\Phi_t(\theta) = \exp[-\lambda \int_{\tau}^{t} (1-e^{i\psi\theta e^{-Bv}})dv]. \tag{56}$$

It is easy to show

$$\lim_{t\to\infty} \Phi_t(\theta) = \Phi(\theta) = \exp[-\lambda \int_{\tau}^{\infty} (1-e^{i\psi\theta e^{-Bv}})dv] \tag{57}$$

is continuous in θ. Thus, the steady-state distribution exists and has characteristic function (57).

Equation (44) allows us to write the capital to labor ratio as

$$k = \exp\ [\frac{A-B-r}{B}]\ , \tag{58}$$

which implies the value of the risky asset $P = \gamma k$ is[23]

$$P = \gamma\exp[\frac{A-B-r}{B}]. \tag{59}$$

Hence, because r converges in distribution, k and P also have steady-state distributions.

Finally, we compare the decentralized solution to a stochastic centralized Ramsey problem. The Ramsey problem for this example is

$$\text{Max } E_0 \int_0^{\infty} U(c(t),t)dt \tag{60}$$

subject to the capital accumulation equation (50). The question is whether the aggregate per capita consumption in the decentralized problem, $c(k) = \rho(1+\gamma)k$, is the solution to (60) for some function $U(c,t)$.

By stochastic dynamic programming, we prove $\rho(1+\gamma)k$ solves the Ramsey problem with

$$U(c,t) = e^{-\varepsilon t}\log c \tag{61}$$

and

$$\varepsilon = \rho(1+\gamma) - B, \tag{62}$$

if $\varepsilon > 0$. As in Theorem 1, define

$$J(k,t) = \operatorname{Max} E_t \int_t^{\infty} e^{-\varepsilon s} \log c(s) ds \tag{63}$$

$$I(k) = e^{\varepsilon t} J(k,t). \tag{64}$$

The Bellman equation for this problem is

$$0 = \operatorname*{Max}_{c} [\log c - \varepsilon I + I_k[(A - B \log k)k - \beta k - c] + \lambda[I(k-\sigma k) - I(k)]]. \tag{65}$$

The optimal consumption rule satisfies

$$c^* = 1/I_k. \tag{66}$$

The function I satisfying the partial differential-difference equation derived when c* is substituted into (65) is

$$I(k) = \frac{\log k}{\varepsilon + B} + a, \tag{67}$$

where

$$a = \frac{1}{\varepsilon} [-\log(\varepsilon+B) - 1 + (A - \beta + \lambda \log(1-\sigma))/(\varepsilon+B)]. \tag{68}$$

Thus, the optimal consumption is $(\varepsilon+B)k = \rho(1+\gamma)k$, the decentralized solution. In the sense of solving (60), the decentralized soluion may be called an efficient path of capital accumulation.

8. Conclusion

We have shown how the rate of aggregate capital accumulation is determined in a stochastic model with decentralized decision-making by households. Individuals solve an intertemporal consumption-portfolio problem and thus determine the aggregate rate of consumption and the demand for capital assets. In our model, there were two assets, riskless capital and a risky asset representing claims to future wage income. In addition, the analysis provided a set of equations for determining the value of the risky asset and these were solved explicitly in an example.

There are several areas for future research. One possibility is to study models in which the stochastic processes governing lifetimes are specified so that different generations are explicitly recognized and more detailed life-cycle behavior is exhibited. Also, general results on the existence of the steady-state are important. Finally, there are the interesting questions of efficiency and the relationships between centralized and decentralized allocation models.

Footnotes

1. It is possible to assume $L(t)$ represents only the labor force and not the total population. Births and deaths then correspond to entering leaving the labor force.

2. Merton [17] discusses a similar model with a continuous-time diffusion approximation to the discrete-time process.

3. It will be convenient throughout the paper to perform an analysis first in discrete-time and then to take limits as the time interval h goes to 0.

4. McKean [14] discusses diffusion processes and Kushner [11] and Feller [9] consider both diffusion and Poisson-driven processes. See Cox and Miller [5] for a less formal treatment of both types of process. Economic applications are presented in Merton [15, 16, 17, 18].

5. Itô's lemma is the stochastic analog in the theory of diffusion processes of the Fundamental Theorem of Calculus. A similar result applies to Poisson-driven processes. See the references in footnote 4 for further discussion.

6. It is assumed there are no alternatives to employing capital and labor in the given technology; i.e. there is no storage of capital and no leisure choice for labor.

7. As (1) shows, $L(t+h)$ is random and consequently the future paths of $K(t)$ and $L(t)$ are uncertain.

8. In addition, of course, he may receive an inheritance at birth.

9. It is important to recognize that the financial assets traded are not uniquely determined by the production technology, wealths, and preferences, but rather the financial structure we discuss represents one possible method for dealing with time and uncertainty. Moreover, even assets with equivalent return structures may differ with respect to price per unit of the asset and number of units (shares) outstanding. For example, one firm may pay dividends by distributing capital to shareholders, while another may equivalently repurchase shares.

10. The following conditions are assumed to hold in this market: there are no transaction costs or taxes, each individual believes he can buy or sell as much of an asset as he wants at the market price, and trading takes place continuously in time.

11. Human capital and physical capital are fundamentally distinguished by the issues of moral hazard and incentives. Because a major purpose of this paper is to study relationships between real and financial assets,

the problems of moral hazard and incentives which arise when an individual agrees to trade away his future wages are not discussed. That is, we do not consider, for example, whether or not individuals work as diligently when they have exchagned future wages for another asset as when they have not so traded.

12. The second term may be considered an insurance premium individuals pay so that shares representing claims on an infinite wage stream may be issued.

13. Ignoring the variance of deaths is consistent with the approximations in (3) and (4).

14. Merton [15] provides a derivation of this equation.

15. The term individual is used to denote either a single person or a family which shares its wealth equally among children, laborers, and retired laborers, and maximizes the utility of overall consumption. In place of an uncertain lifetime and bequest function, the infinite horizon problem with discounting is solved. We note that in order to analytically solve the problem with uncertain, finite lifetime, assumptions are often made to reduce the problem to an infinite horizon problem. In Cass and Yaari [4] and Merton [15], for example, an exponential lifetime problem is so reduced.

16. See Kushner [11] for a proof and Merton [15] for a discussion in the portfolio context.

17. Equation (29) implies the aggregate consumption rate c is linear in national wealth $W = \Sigma_i W^i$. Hence for certain parameter and wealth values, c may exceed the output rate f(k). Some growth models constrain consumption so that $c \leq f(k)$. If this constraint must always hold in the decentralized model, then output becomes distinguished from capital in place and a separate price must be established to allocate output. In such a situation one capital share would not always be equivalent to one unit of output.

18. See Merton [15] for a discussion of the HARA class of utility functions.

19. Note that if $c = \rho(k+P(k))$ is substituted into (12), then k is a time homogeneous Markov process. Thus, there is consistency between the assumed time homogeneity and Markov property of the P process and the dynamics of k which depends on P through c and upon which P depends.

20. This production function exhibits capital saturation at $\exp((A-B)/B) \equiv k_s$.

21. The variable Z is $\alpha_{ll} - (r-\delta)$. If $\gamma > 0$, i.e. the risky asset has a positive value, then it is easy to show $Z > 0$.

22. If $B-\beta-\rho(1+\gamma) > 0$, then for any trajectory with initial point, $k_0 \leq k_s$, the capital saturation point, there is a positive probability that

$k(t) > k_s$ for some t. Furthermore, if no inventories of capital may be held outside the production process, then for $k > k_s$ there will be a negative interest rate r.

For one possible way to deal with this, see the discussion of a monetary policy in Cass and Yaari [4]. Also note that if $\beta > B$, then any trajectory with $k_0 \leq k_s$ will never exceed k_s and therefore our solution for the consumption rate is optimal whether or not inventory holding is possible.

23. Equations (58) and (59) imply r may be chosen as the state variable instead of k.

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